From Turing Morphogenesis to Attractor Programming

Bifurcation theory and computer science

Alan Heirich 07 November 2024

Turing Morphogenesis

How did the tiger get its stripes?

$$\partial_t \boldsymbol{q} = \underline{\underline{\boldsymbol{D}}}\,\nabla^2 \boldsymbol{q} + \boldsymbol{R}(\boldsymbol{q}),$$

- Turing starts with Reaction-Diffusion systems
- Equations coupled with a solver imply a computer program
- This program “prints” spatial patterns such as the stripes of a tiger, spots of a leopard, and many other “Turing patterns”
- This remains a dominant biological theory today
- Explanation depends on bifurcation theory

Bifurcation theory

The relationship between quantitative and qualitative

- Originates with work of Poincare (1885)
- Explains how small quantitative changes in a nonlinear system can lead to large qualitative changes in behavior of that system
- Basis of a theory of Qualia
- Defines changes in the Eigenspectrum of the Jacobian matrix of first partial derivatives leading to changes in the solution topology
- Extensively used in theory of chaos, but we are interested in the non-chaotic regime

Reverse engineering a dynamical system

From fixed points to equations

- Elementary bifurcation theory describes precise conditions for the creation or destruction of fixed point solutions in a dynamical system
- This process can be reverse engineered to construct a dynamical system with a desired set of fixed point solutions
  - Computer algebra can modify the Jacobian matrix to produce the desired result
  - Construct the governing equations from the modified Jacobian
  - This process can be automated

Reverse Engineering a Dynamical System

From generalized attractors to equations

- Instead of fixed point solutions consider generalized attractors F(X)=0
- Want to reverse engineer a dynamical system with that attractor
- Elementary bifurcation theory is inadequate because it deals with points
    - Need to deal with functions F(X)=0

Test case - Quadratic assignment

Without loss of generality

- Consider the quatratic assignment problem: $\sum_{a,b \in P} w(a,b) \cdot d(f(a), f(b))$
- This is an optimization problem minimize F(x).
    - Convert to a decision problem F(X)=k for decreasing k
- Construct a dynamical system that converges to solutions of the decision problem

Attractor Programming

A summary applied to quadratic assignment

- Formulate an attractor F(X)=0
  - K = $\sum_{a,b \in P} w(a,b) \cdot d(f(a), f(b))$ for quadratic assignment, e.g.
- Posit a generic dynamical system, e.g. $\nabla^2 f = 0$
- Modify the Jacobian matrix of the dynamical system in order to attract it to the attractor
  - Use computer algebra to modify the Eigenspectrum
  - This process can be automated
- Simulate the resulting dynamical system. It will converge to the attractor from any initial conditions within the basin of attraction. The result is a program that solves e.g. quadratic assignment